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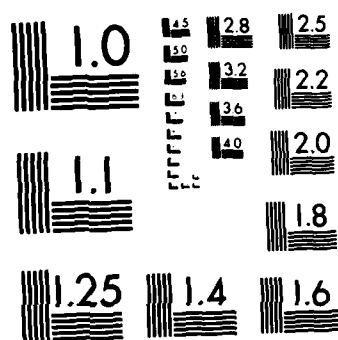
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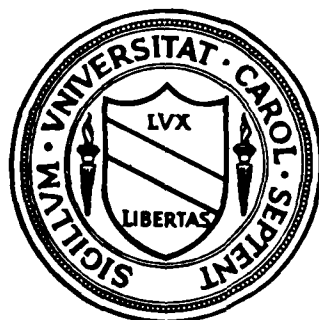
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## CENTER FOR STOCHASTIC PROCESSES

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The Symmetry Group and Exponents of  
Operator Stable Probability Measures

by

William N. Hudson,

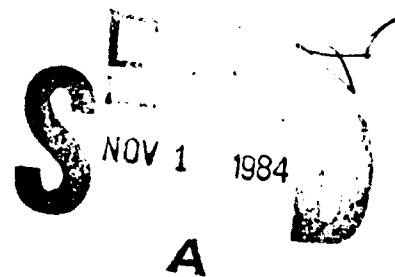
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# THE SYMMETRY GROUP AND EXPONENTS OF OPERATOR STABLE PROBABILITY MEASURES

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## Abstract

There exist exponents of an operator stable measure which commute with every operator in the measure's symmetry group. These exponents together with a new norm lead to some simplifications in the representation of the Lévy measure.

Keywords: Operator-stable laws, multivariate symmetric stable distributions, multivariate stable laws.

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## 0. Introduction

An operator-stable (OS) probability measure  $\mu$  on a normed finite-dimensional real vector space  $V$  is the limit distribution of operator normed and centered sums of a sequence of i.i.d. random vectors in  $V$ . The classical stable laws on  $\mathbb{R}^1$  are a special case. If  $\mu$  is full and operator stable, then  $\mu$  is infinitely divisible so if  $\hat{\mu}$  is the ch.f. of  $\mu$ , then for  $t > 0$ ,  $\hat{\mu}^t$  is the ch.f. of an infinitely divisible measure  $\mu^t$ . The role of the index in the one-dimensional case is played by an invertible linear operator  $B$  on  $V$  called the exponent of  $\mu$ . If we define  $t^B = \exp\{(\ln t)B\} = \sum_{j=0}^{\infty} \frac{(\ln t)^j}{j!} B^j$ , then  $B$  is an exponent for  $\mu$  if

$$(1) \quad \mu^t = t^B \mu * \delta(b(t)), \quad t > 0,$$

where  $\delta(b(t))$  is the unit mass at  $b(t) \in V$  and  $t^B \mu = \mu t^{-B}$ . In [7] it was proved that full OS distributions always have at least one exponent.

An exponent of a full OS law  $\mu$  determines much of its structure. (See [2] and [7] for the results which are now described.) In general  $\mu$  has both a Gaussian component  $\mu_g$  and a Poisson component  $\mu_p$ . These components are concentrated on independent subspaces determined by the exponent  $B$ . To be precise let  $f(x)$  denote the minimal polynomial of  $B$ . Then  $f(x) = g(x)h(x)$  where the roots of  $g$  have real parts equal to  $\frac{1}{2}$  while those of  $h$  have real parts greater than  $\frac{1}{2}$ . The Gaussian component  $\mu_g$  is concentrated on  $V_g = \text{kernel}(g(B))$  while  $\mu_p$  is concentrated on  $V_p = \text{kernel}(h(B))$ . Furthermore  $V = V_g \oplus V_p$ ,  $\mu_g$  and  $\mu_p$  are full and OS on  $V_g$  and  $V_p$  respectively. The exponents of  $\mu_g$  and  $\mu_p$  are the restrictions of  $B$  to  $V_g$  and  $V_p$  respectively. Now let  $M$  denote the Lévy measure of  $\mu$ . The exponent determines a major part of the structure of  $M$ . From (1) upon noting that  $t \cdot M$  is the Lévy measure of  $\mu^t$  and that  $t^B M = M t^{-B}$  is the Lévy measure of  $t^B \mu$ , one sees that  $t \cdot M = t^B M$ . This fact can be used to show that if  $A$  is a Borel subset of  $V_p$ , then

$$(2) \quad M(A) = \int_L M_x(A) K(dx)$$

where  $K$  is a finite measure on a Borel subset  $L$  of the unit sphere  $U$  in  $V_p$  and  $M_x$  is concentrated on the single orbit  $\{t^B x: t > 0\}$  determined by  $x$ . The Lévy measure  $M_x$  also satisfies the condition that  $t \cdot M_x = t^B M_x$  and as a result,

$$M_x\{t^B x: t > s\} = 1/s, \quad s > 0,$$

(i.e.  $M_x(A) = \int_0^\infty I_A(t^B x) t^{-2} dt$ ). From (2) it follows that the support of  $M$  is the union of orbits of  $t^B$ . Each orbit begins at the origin and extends to infinity (i.e.  $\lim_{t \rightarrow 0} t^B x = 0$  and  $\lim_{t \rightarrow \infty} \|t^B x\| = \infty$ ). The shape of these orbits is determined by the exponent  $B$ . In particular cases orbits can be straight lines ( $B = \lambda I$ ), half of a parabola ( $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $V = \mathbb{R}^2$ ), or spirals (e.g.  $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + Q$  where  $Q + Q^* = 0$  so  $t^Q$  is a rotation). The expression for  $M_x$  above shows that the tail behavior of  $M$  along orbits is determined by  $B$ . The measure  $K$  assigns weights to the orbits and determines which orbits are included in the support of  $M$ . Together  $B$  and  $K$  determine  $M$ . But, in general  $B$  and  $K$  are not unique. Is there a reasonable way to choose a particular exponent and measure  $K$ ? The set of exponents depends on the amount of symmetry possessed by  $\mu$ . Call a linear operator  $A$  on  $V$  a symmetry of  $\mu$  if for some  $a \in V$ ,  $\mu = A\mu * \delta(a)$ . It is natural to expect that a symmetry of  $\mu$  should take orbits into orbits while leaving  $K$  invariant. (See Theorem 7 below.) In particular, if  $BA = AB$ , then  $At^B x = t^B Ax$  (since  $t^B$  is a power series in  $B$ ) so orbits are taken by  $A$  into orbits. Furthermore the requirement that  $B$  commutes with every symmetry tends to pick out exponents with nice properties whenever possible. (See Theorems 4 and 5.)

Example. Suppose that  $\mu$  is the standard Gaussian measure on  $\mathbb{R}^d$ . If  $X$  and  $Y$  are i.i.d.  $\mu$ , the measure corresponding to  $X+Y$  is  $\mu * \mu = 2^{-1/2} \mu$ . One suspects (and easily verifies) that  $\frac{1}{2}I$  is an exponent for  $\mu$ . Suppose that  $S$  is a skew operator, that is, that  $S + S^* = 0$ . For each  $t > 0$ ,  $t^S$  is orthogonal and so  $t^S \mu = \mu$ , i.e.  $t^S$  is a symmetry of  $\mu$ . It follows that  $\frac{1}{2}I + S$  is also an exponent for  $\mu$ , for any skew operator  $S$  (see Theorem 1 below). Thus operator stable measures may have many

exponents; the number of exponents depends on the size of the collection of symmetries of  $\mu$ . Does an operator stable measure have a "simplest" exponent?

A lemma of Schur's ([6], p. 173) suggests a possible answer. This lemma says: "Let  $F$  be a family of linear operators on a Hilbert Space  $H$  and suppose that the only closed subspaces which are invariant under every operator in  $F$  are  $\{0\}$  and  $H$ . If  $A$  is a self-adjoint linear operator on  $H$  that commutes with every operator in  $F$ , then  $A=cI$  for some scalar  $c$ ." (As usual,  $I$  denotes the identity operator.) Schur's Lemma suggests that the "simplest" exponent would be one which commutes with a large collection of operators. In this example,  $\frac{1}{2}I$  is the only exponent of  $\mu$  which commutes with every symmetry of  $\mu$ . We will show below that there is always an exponent of  $\mu$  which commutes with all the symmetries of  $\mu$ . (Theorem 2)

Our results on commuting exponents are applied to simplify the representation of the Lévy measure of an OS law in section 3. There we define a new norm. The unit sphere relative to this norm plays the role of  $L$  above. The corresponding mixing measure  $K$  does not depend on the choice of an exponent (Theorem 6). This representation provides a simple relationship between the symmetries of  $\mu$  and those of  $K$ . These results complement those of Kucharczak [5], Jurek [3], and Hudson-Mason [2].

### 1. Preliminaries

Let  $\mu$  be a full OS probability measure on a finite dimensional real vector space  $V$ .  $GL(V)$  denotes the set of all invertible operators on  $V$ . For  $A \in GL(V)$ , we define  $A\mu = \mu \circ A^{-1}$ . Two groups of interest in connection with  $\mu$  are the symmetry group

$$S(\mu) = \{A \in GL(V) : A\mu * \delta(a) = \mu \text{ for some } a \in V\}$$

and

$$G = \{A \in GL(V) : \text{for some } t > 0, \text{ for some } a \in V, \mu^t = A\mu * \delta(a)\}.$$



It is known that  $S(\mu)$  is a compact, normal subgroup of  $G$ . For any closed group  $H$ ,  $TH$  will denote the tangent space of  $H$ . Thus  $A \in TH$  if and only if  $A = \lim_{n \rightarrow \infty} (H_n - I)/d_n$  where  $\{H_n\} \subset H$  and  $\{d_n\}$  is a real null sequence. We recall that the exponential maps  $TH$  onto the connected component of  $I$  in  $H$ .  $CH$  will denote the center of  $H$ , that is, those elements of  $H$  which commute with every element of  $H$ . Recall that  $CH$  is a subgroup of  $H$ .

The collection of exponents of  $\mu$ , denoted  $E(\mu)$ , is the set of all operators for which (1) holds. The following result gives a basic fact about exponents.

Theorem 1. Let  $B \in E(\mu)$ . Then

- (i) Every eigenvalue of  $B$  has real part  $\geq \frac{1}{2}$ ,
- (ii)  $E(\mu) = B + TS(\mu)$ .

For a proof of this result see [1] and [7].

## 2. Commuting Exponents

In this section we investigate the existence of an exponent which commutes with every operator in  $S(\mu)$ . Such exponents will be called commuting and the collection of commuting exponents will be denoted by  $E_c(\mu)$ .

Proposition 1. Let  $A \in S(\mu)$  and  $B \in E(\mu)$ . Then  $ABA^{-1} \in E(\mu)$ . Moreover, if  $S(\mu)$  is discrete, the unique exponent  $B$  is commuting.

Proof. We have  $A\mu = \mu * \delta(a)$  and

$$(A\mu)^t = A\mu^t = A(t^B \mu * \delta(b(t))) = At^B \mu * \delta(Ab(t)) = t^{ABA^{-1}} (A\mu) * \delta(Ab(t)).$$

Hence

$$\mu^t = t^{ABA^{-1}} \mu * \delta(Ab(t) - ta + t^{ABA^{-1}} a)$$

and  $ABA^{-1} \in E(\mu)$ . Now if  $S(\mu)$  is discrete,  $TS(\mu) = 0$  and  $B$  is the unique exponent.

Thus  $ABA^{-1} = B$  and  $B$  is commuting.

(QED)

The following example shows that not all exponents are commuting.

Example. Let  $\mu$  be the symmetric Cauchy distribution on  $\mathbb{R}^2$ . Then  $I \in E(\mu)$  and  $S(\mu)$  is the full orthogonal group. Hence  $TS(\mu)$  consists of the skew symmetric operators. By Theorem 1,  $E(\mu) = I + TS(\mu)$  so  $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$  is an exponent. Also  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in S(\mu)$ . A direct computation shows that  $AB \neq BA$ . Furthermore,  $A$  does not map orbits into other orbits.

The main result of this section is that commuting exponents always exist.

Theorem 2.  $E_c(\mu)$  is non-empty.

Proof. Let  $H$  be a Haar probability measure on the compact group  $S(\mu)$ , and let  $B \in E(\mu)$ . Define

$$M = \int_{S(\mu)} sBs^{-1} dH(s).$$

Since  $E(\mu)$  is closed and convex by Theorem 1 and closed under conjugation by elements of  $S(\mu)$  by Proposition 1,  $M \in E(\mu)$ . If  $A \in S(\mu)$ , then by the invariance property of Haar measure

$$AMA^{-1} = \int_{S(\mu)} AsBs^{-1}A^{-1} dH(s) = \int_{S(\mu)} (As)B(As)^{-1} dH(s) = \int_{S(\mu)} sBs^{-1} dH(s) = M.$$

Thus  $M \in E_c(\mu)$ .

(QED)

The collection of all commuting exponents is characterized in our next result.

Theorem 3. Suppose  $B \in E_c(\mu)$ . Then  $E_c(\mu) = B + TCS(\mu)$ .

Proof. Assume  $B \in E_c(\mu)$ . Using the relation between groups and their tangent spaces one readily verifies the equivalence of the following statements.

- (i)  $\tilde{B} \in E_c(\mu)$ ,
- (ii)  $\tilde{B} - B \in TS(\mu)$  and  $\tilde{B} - B$  commutes with every element of  $S(\mu)$ .
- (iii) For all  $t$ ,  $\exp\{t(\tilde{B} - B)\} \in CS(\mu)$ , and
- (iv)  $\tilde{B} - B \in TCS(\mu)$ .

(QED)

Corollary.  $E_c(\mu) = E(\mu)$  if and only if  $TS(\mu) = TCS(\mu)$ .

We now examine the extent to which the structure of a commuting exponent is determined by the "size" of  $S(\mu)$ .

Theorem 4. Let  $B \in E_c(\mu)$ . If the only proper subspace of  $V$  invariant under  $S(\mu)$  is 0, then  $B = \lambda I + WQW^{-1}$ , where  $W$  is positive definite and  $Q$  is skew symmetric. Furthermore either  $Q = 0$  or  $Q^2 = WQ^2W^{-1} = -\beta^2 I$  for some  $\beta > 0$ .

Proof. Since  $S(\mu)$  is compact, there is a positive definite operator  $W$  and a closed subgroup  $G$  of the orthogonal group such that

$$S(\mu) = WGW^{-1}.$$

It follows that  $S(W^{-1}\mu) = G$ . Since  $B \in E_c(\mu)$ ,  $B_0 \equiv W^{-1}BW \in E_c(W^{-1}\mu)$ . Write  $B_0 = B_1 + B_2$  where  $B_1 = \frac{1}{2}(B_0 + B_0^*)$  is self-adjoint and  $B_2 = \frac{1}{2}(B_0 - B_0^*)$  is skew-symmetric. Since  $B_0 \in E_c(W^{-1}\mu)$ ,  $AB_0 = B_0A$  for  $A \in G$ . Take adjoints to see that  $B_0^*A^* = A^*B_0^*$  for  $A \in G$ . But every operator in  $G$  is orthogonal so  $G = \{A^*: A \in G\}$ . Thus

$$AB_0^* = B_0^*A, \quad A \in G.$$

It follows that every operator in  $G$  commutes with  $B_1$  which is self-adjoint. Now by hypothesis the only proper subspace of  $V$  invariant under  $S(\mu)$  and hence under  $G$  is 0. By Schur's Lemma,  $B_1 = \lambda I$  for some real number  $\lambda$ . Now consider  $B_2$ . Since  $B_2$  is skew-symmetric, it is normal and thus its minimal polynomial is the product  $p_1(x), \dots, p_k(x)$  of distinct irreducible polynomials. If  $k > 1$ , then  $\ker p_1(B_2)$  is a proper subspace of  $V$  which is invariant under  $G$  contrary to our hypothesis. Thus  $k = 1$  and the minimal polynomial of  $B_2$  is either  $x$  or  $x^2 + \beta^2$  for some  $\beta > 0$ . (A skew-symmetric operator has purely imaginary eigenvalues). If it is  $x$ , then  $B_2 = 0$ ; otherwise,  $B_2^2 = -\beta^2 I$ . From  $B_0 = B_1 + B_2 = \lambda I + B_2$ , we obtain upon setting  $Q = B_2$ ,

$$W^{-1}BW = \lambda I + Q$$

or  $B = \lambda I + WQW^{-1}$ . Finally  $B \in E(\mu)$  so the real part of every eigenvalue of  $B$  is not less than  $\frac{1}{2}$ , i.e.  $\lambda \geq \frac{1}{2}$ . (QED)

Corollary. If in addition to the hypothesis of the theorem, either  $B$  is diagonalizable or  $\dim V$  is odd, then  $B = \lambda I$ .

Proof. First suppose  $B$  is diagonalizable. Let  $v$  be an arbitrary eigenvector of  $B$  so  $Bv = \lambda_0 v$  for some real  $\lambda_0$ . By Theorem 4,  $B = \lambda I + WQW^{-1}$  so  $v$  is an eigenvector of  $WQW^{-1}$ . In particular  $WQW^{-1}v = (\lambda_0 - \lambda)v$ . Hence  $(WQW^{-1})^2 v = (\lambda_0 - \lambda)^2 v$ . But  $WQW^{-1} = 0$  or  $(WQW^{-1})^2 = -\beta^2 I$ . In either case it follows that  $\lambda_0 = \lambda$ . Since  $B$  is diagonalizable,  $B = \lambda I$ . Now suppose  $\dim V$  is odd. Since  $Q$  is skew symmetric,

$$\det Q = \det Q^* = \det(-Q) = -\det Q,$$

so  $Q$  is singular. Hence  $Q^2 \neq -\beta^2 I$  and therefore  $Q = 0$ .

(QED)

A slight refinement of the preceding theorem is given in

Theorem 5. Suppose  $B \in E_c(\mu)$  has  $p$  real eigenvalues  $\lambda_1, \dots, \lambda_p$  with corresponding eigenvectors  $v_1, \dots, v_p$ . If  $\{Av_i : A \in S(\mu) \ 1 \leq i \leq p\}$  spans  $V$ , then  $B$  is diagonalizable with spectrum  $\{\lambda_1, \dots, \lambda_p\}$ . Thus if  $\lambda_1 = \dots = \lambda_p = \lambda$ ,  $B = \lambda I$ .

Proof. For  $A \in S(\mu)$ ,  $BAv_i = ABv_i = \lambda_i Av_i$ , so  $Av_i$  is an eigenvector of  $B$  with eigenvalue  $\lambda_i$ . Hence there is a basis of  $V$  consisting of eigenvectors of  $B$  and so  $B$  is diagonalizable.

(QED)

Corollary. In  $R^2$  if  $B \in E_c(\mu)$  and if there is a reflection  $A \in S(\mu)$ , then  $B$  is self-adjoint.

Proof. Select orthonormal vectors  $v_1$  and  $v_2$  so that  $Av_1 = v_1$  and  $Av_2 = -v_2$ . Then  $ABv_1 = Bv_1$  and  $ABv_2 = -Bv_2$ , so  $Bv_1 = \lambda_1 v_1$  and  $Bv_2 = \lambda_2 v_2$  where  $\lambda_1$  and  $\lambda_2$  are real.

(QED)

### 3. The Lévy measure

In this section we discuss the relationship between commuting exponents and the representation of the Lévy measure of  $\mu$ . Since  $\mu$  is infinitely divisible, one can write the characteristic function of  $\mu$  in the canonical form

$$\hat{\mu}(y) = \exp\{i\langle y, a \rangle - \frac{1}{2}\langle y, \Sigma y \rangle + \int \psi(x, y) M(dx)\}$$

where  $a \in V$ ,  $\Sigma$  is a non-negative definite self-adjoint operator,  $M$  is a

$\sigma$ -finite measure satisfying

$$\int_V ||x||^2 \wedge 1M(dx) < \infty,$$

and

$$\psi(x,y) = \exp\{i\langle x,y \rangle\} - 1 - \frac{i\langle x,y \rangle}{1+\langle x,x \rangle}.$$

For OS measures it has been shown that one can further decompose the Lévy measure  $M$  as follows. For an exponent  $B$  of  $\mu$  set  $L_B = \{x: ||x||=1 \text{ and } ||t^B x|| > 1 \text{ for all } t > 1\}$  and define the mixing measure  $K_B$  on the Borel subsets  $A$  of  $L_B$  by

$$K_B(A) = M(\{t^B x: x \in A, t \geq 1\}).$$

Thus  $K_B$  assigns mass to the particular orbits  $\{t^B x: t > 0\}$ . Note that both  $L_B$  and  $K_B$  depend on the choice of exponent  $B$ . In terms of  $K_B$  the Lévy measure  $M$  is given by

$$(3) \quad M(S) = \int_{L_B} \int_0^\infty I_S(t^B x) t^{-2} dt dK_B(x)$$

(See [2] and [3].) It was necessary to introduce the subset  $L_B$  of  $U$  since for some exponents, orbits may intersect the unit sphere more than once.

We now introduce a new norm  $|||\cdot|||$  which depends on the particular OS law but not on the choice of exponent. The unit sphere  $U' = \{v: |||v|||=1\}$  induced by this norm will intersect each orbit once and so may play the role of  $L_B$ . As above we define a mixing measure  $K$  on the Borel subsets  $A$  of  $U'$  by  $K(A) = M(\{t^B x: x \in A, t \geq 1\})$ . This measure  $K$  also does not depend on the choice of exponent and the representation (3) of the Lévy measure  $M$  in terms of  $K$  is still valid. The new norm leads to a system of "polar" coordinates with nice properties. (cf. Jurek [4]).

For  $x \in V$ , and  $B \in E(\mu)$  define  $|||x||| = \int_0^1 \int_{S(\mu)} ||t^B x|| H(dg) t^{-1} dt$  where  $H$  again denotes Haar measure on  $S(\mu)$  and  $||\cdot||$  is the original norm on  $V$ .

Proposition 3. If  $\mu$  is full and OS on  $V$ , then

(i)  $|||\cdot|||$  does not depend on the choice of  $B \in E(\mu)$ ,

(ii)  $|||\cdot|||$  is a norm on  $V$ ,

(iii) for  $A \in S(\mu)$ ,  $|||Ax||| = |||x|||$ ,

(iv)  $t \mapsto |||t^B x|||$  is strictly increasing on  $(0, \infty)$  for each  $x \neq 0$ , and

- (v) the map  $\phi_B: U'x(0, \infty) \rightarrow V \setminus \{0\}$  defined by  $\phi_B(x, t) = t^B x$  is a homeomorphism when  $U'x(0, \infty)$  has the product topology.

Proof. (i). Let  $B \in E(\mu)$  and let  $B_0 \in F_c(\mu)$ . By Theorem 1,  $B - B_0 \in TS(\mu)$  so for all  $t > 0$ ,  $B_0 t^{B-B_0} = t^{B-B_0} B_0$ . Differentiate to see that  $B_0$  commutes with  $B - B_0$  and consequently that  $t^B = t^{B-B_0} t^{B_0}$ . For  $x \in V$ , use the invariance property of Haar measure to obtain the equalities

$$\begin{aligned} |||x|||_B &= \int_0^1 \int_{S(\mu)} ||gt^B x|| t^{-1} H(dg) dt \\ &= \int_0^1 \int_{S(\mu)} ||gt^{B-B_0} t^{B_0} x|| t^{-1} H(dg) dt = |||x|||_{B_0}. \end{aligned}$$

This proves (i) and allows us to omit the subscript  $B$ .

(ii) This is obvious.

(iii) Let  $A \in S(\mu)$ . By (i)

we may assume that  $B \in E_c(\mu)$ . Then

$$\begin{aligned} |||Ax||| &= \int_0^1 \int_{S(\mu)} ||gt^B Ax|| t^{-1} H(dg) dt \\ &= \int_0^1 \int_{S(\mu)} ||gAt^B x|| t^{-1} H(dg) dt = |||x|||. \end{aligned}$$

(iv) Suppose that  $0 < r < s$ . Then

$$\begin{aligned} |||r^B x||| &= \int_0^1 \int_{S(\mu)} ||g(tr)^B x|| t^{-1} H(dg) dt \\ &= \int_0^r \int_{S(\mu)} ||gu^B x|| u^{-1} H(dg) du \\ &< \int_0^s \int_{S(\mu)} ||gu^B x|| u^{-1} H(dg) du = |||s^B x|||. \end{aligned}$$

(v) It follows from (iv) that  $\phi_B$  is one-to-one. Since every point in  $V \setminus \{0\}$  lies on some orbit,  $\phi_B$  is "onto". The continuity of  $\phi_B$  is well-known and easily checked. To show  $\phi_B^{-1}$  is continuous write  $\phi_B^{-1}(x) = (\ell(x), \zeta(x))$  so that  $\ell(x) \in U'$ ,  $\zeta(x) > 0$  and  $x = \zeta(x)^B \ell(x)$ . Suppose that  $x_n \rightarrow x \neq 0$ . Assume some subsequence

$\ell(x_n)$  tends to infinity. Then since the eigenvalues of  $B$  have positive real parts,  $\|x_n\| = \|\zeta(x_n)^B \ell(x_n)\| \rightarrow \infty$  contrary to the convergence of  $x_n$ . It follows that  $(\ell(x_n), \zeta(x_n))$  is a bounded sequence in  $U' \times (0, \infty)$ . Let  $(\ell(x_{n_k}), \zeta(x_{n_k}))$  be any convergent subsequence and let  $(\ell_0, \zeta_0) = \lim(\ell(x_{n_k}), \zeta(x_{n_k}))$ . Then

$$x = \lim x_n = \lim \zeta(x_n)^B \ell(x_n) = \zeta_0^B \ell_0.$$

Since  $\phi_B$  is one-to-one,  $\zeta(x) = \zeta_0$ , and  $\ell(x) = \ell_0$ . Thus every convergent subsequence of  $(\ell(x_n), \zeta(x_n))$  has the same limit, namely  $(\ell(x), \zeta(x))$ . This proves that  $\phi_B^{-1}$  is continuous. (QED)

The proof that  $\phi_B^{-1}$  is continuous was given above for the sake of completeness, c.f. [4].

Part (ii) of Proposition 3 implies that each orbit intersects  $U'$  exactly once. The fact that  $U'$  is closed and that  $\phi_B$  is a homeomorphism is useful in proving weak convergence results.

**Theorem 6.** Let  $\mu$  be full OS with Lévy measure  $M$  and let  $B \in E(\mu)$ . Let  $F$  and  $E$  be any Borel subsets of  $V \setminus \{0\}$  and  $U'$  respectively. Then

$$(4) \quad M(F) = \int_{U'} \int_0^\infty I_F(s^B x) s^{-2} ds K(dx)$$

where  $K$  is a finite Borel measure on  $U'$  and

$$(5) \quad K(E) = M\{t^B x : x \in E, t \geq 1\}$$

The measure  $K$  does not depend on the choice of  $B \in E(\mu)$ .

**Proof.** The proof of (4) and (5) is similar to that of (3) in [2] or [3] and is therefore omitted.

The proof that  $K$  does not depend on the choice of exponent will involve an easy lemma.

**Lemma 3.1** Let  $g \in S(\mu)$  and  $B \in E(\mu)$ . If  $gB = Bg$ , then  $gK_B = K_B$ .

Proof. Let  $D$  be any Borel subset of  $U'$ . Then

$$\begin{aligned}
 gK_B(D) &= K_B(g^{-1}(D)) \\
 &= M\{t^B x: x \in g^{-1}(D), t \geq 1\} \\
 &= M\{t^B g^{-1}x: x \in D, t \geq 1\} \\
 &= M(g^{-1}\{t^B x: x \in D, t \geq 1\}) \\
 &= (gM)(\{t^B x: x \in D, t \geq 1\}).
 \end{aligned}$$

But  $g \in S(\mu)$  and hence  $gM = M$ . Thus

$$gK_B(D) = M\{t^B x: x \in D, t \geq 1\} = K_B(D). \quad (\text{QED})$$

Now let  $A$  be any Borel subset of  $V \setminus \{0\}$ . Then if  $B \in E(\mu)$

$$\begin{aligned}
 M(A) &= \int_0^\infty \int_{U'} I_A(t^B x) t^{-2} K_B(dx) dt \\
 &= \int_0^\infty K_B((t^{-B}A) \cap U') t^{-2} dt.
 \end{aligned}$$

Let  $B_0 \in E_c(\mu)$ . It suffices to prove that  $K_B = K_{B_0}$ . So let  $D$  be any Borel subset of  $U'$ , and put  $A = \{s^B x: x \in D, s \geq 1\}$ . Then



$$(t^{-B}A) \cap U' = \begin{cases} \phi & \text{if } t < 1 \\ D & \text{if } t \geq 1. \end{cases}$$

Hence

$$K_B(D) = M(A) = \int_0^\infty K_{B_0}((t^{-B_0}A) \cap U') t^{-2} dt.$$

But  $B_0 \in E_c(\mu)$  so  $B_0$  commutes with  $B-B_0$ . Furthermore  $t^{B-B_0} \in S(1)$

and  $t^{B-B_0}U' = U'$ . It follows from Lemma 3.1 that

$$\begin{aligned} K_{B_0}((t^{-B}A) \cap U') &= K_{B_0}((t^{B-B_0}(t^{-B_0}A)) \cap U') \\ &= K_{B_0}((t^{-B_0}A) \cap U'). \end{aligned}$$

Therefore,

$$\begin{aligned} K_B(D) &= \int_0^\infty K_{B_0}((t^{-B}A) \cap U') t^{-2} dt. \\ &= \int_1^\infty K_{B_0}(D) t^{-2} dt = K_{B_0}(D). \end{aligned} \quad (\text{QED})$$

Remark. There is a converse to Theorem 6. If  $B$  is an OS exponent, and if  $K$  is a finite Borel measure on  $U'$  of  $V_p$ , then the measure  $M$  defined by

$$M(F) = \int_{U'} \int_0^\infty I_F(s^B x) s^{-2} ds K(dx)$$

is the Lévy measure of some OS law with exponent  $B$ . Again see [2] or [3]. In [7] Sharpe characterized the set of OS exponents, i.e. those operators which are the exponent of some OS law.

We now consider the relationship between  $S(K)$ , the symmetry group of the measure  $K$  in Theorem 6, and  $S(\mu)$ .

Theorem 7. Let  $\mu$  be a full OS measure on  $V$ . Then  $S(\mu) = S(K)$ .

Proof. Let  $A \in S(\mu)$ . Since by Proposition 3,  $\|Ax\| = \|x\|$ ,  $AU' = U'$ . Since  $K$  does not depend on the choice of an exponent, we may assume  $B \in E_c(\mu)$ .

Then  $S(\mu) \subset S(K)$  follows from Lemma 3.3.

(OED)

The following example shows that even if an OS measure  $\mu$  has no Gaussian component, if the original norm on  $V$  is used and if  $M$  is defined as in (3), then  $S(K)$  may be much larger than  $S(\mu)$  even though  $K$  is full. (To see that in this example  $\mu$  has no Gaussian component, note that no eigenvalue of  $B$  has real part equal to  $1/2$ .)

Example. Take  $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ . Then  $L_B$  is the unit circle in  $\mathbb{R}^2$ . Let  $K$  be the Lebesgue measure on the circle. Then  $K$  is full and  $S(K)$  is the orthogonal group. Define  $M$  (and hence  $\mu$ ) in terms of  $K$  and  $B$  using equation (3). Then  $\mu$  is a full OS measure with  $B \in E(\mu)$  (see [2]). We now find  $S(\mu)$ . First note that  $S(\mu)$  is closed and  $V = \mathbb{R}^2$  so if  $S(\mu)$  were not discrete,  $S(\mu)$  would be conjugate to the orthogonal group. Then by Theorem 4,  $B$  would have conjugate complex eigenvalues. Hence  $S(\mu)$  is discrete, and  $B \in E_c(\mu)$  by Proposition 1. Now suppose  $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S(\mu)$ . Then since  $B \in E_c(\mu)$   $BD = DB$  and so  $c = b = 0$ . Since  $S(\mu)$  is a compact group, the fact that  $D^n \in S(\mu)$  for all  $n$  shows  $|a| = |d| = 1$ . A direct computation now shows that  $S(\mu) = S(M) = \{\pm I, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$ .

### References

- [1] Holmes, J.P., W.N. Hudson and J.D. Mason. Operator stable laws: Multiple exponents and elliptical symmetry. *Ann. Probability* 10 No. 3, 602-612 (1982).
- [2] Hudson, W.N. and J.D. Mason. Operator stable laws. *J. Multivariate Anal.* 11 No. 3, 434-447 (1981).
- [3] Jurek, Z.J. Remarks on operator stable probability measures. *Commentationes Mathematicae XXI*, 71-74 (1979).
- [4] Jurek, Z.J. Polar coordinates in Banach spaces. *Bull. Acad. Polon. Sci.* 31 (To appear).
- [5] Kucharczak, J. Remarks on operator stable measures. *Colloq. Math.* 34, 109-119 (1975).
- [6] Lang, S. *Analysis II*, Addison-Wesley, Reading, Mass. (1969).
- [7] Sharpe, M. Operator stable probability distributions on vector groups. *Trans. AMS* 136, 51-65 (1969).

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